Phase-space analysis of interacting phantom cosmology

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We perform a detailed phase-space analysis of various phantom cosmological models, where the dark energy sector interacts with the dark matter one. We examine whether there exist late-time scaling attractors, corresponding to an accelerating universe and possessing dark energy and dark matter densities of the same order. We find that all the examined models, although accepting stable late-time accelerated solutions, cannot alleviate the coincidence problem, unless one imposes a form of fine-tuning in the model parameters. It seems that interacting phantom cosmology cannot fulfill the basic requirement that led to its construction.

I. INTRODUCTION

Recent cosmological observations support that the universe is experiencing an accelerated expansion, and that the transition from the deceleration phase to the accelerated phase happened in the recent past [1]. In order to explain this remarkable phenomena, one can modify the theory of gravity, such as $f(R)$ gravity [2, 3], Dvali-Gabadadze-Porrati model [4], and string inspired models [5]. An alternative direction is to introduce the concept of dark energy which provides the acceleration mechanism. The simplest candidate of dark energy which is consistent with current observations is the cosmological constant. Due to the lack of a good explanation of the small value of the cosmological constant, many dynamical dark energy models were explored, such as a canonical scalar field (quintessence) model [6], a phantom model that has equation of state parameter $w < -1$ [7, 8, 9], or the combination of quintessence and phantom in a unified model named quintom [10]. Moreover, many theoretical studies are devoted to shed light on dark energy within some quantum gravitational principles, such as the so-called "holographic dark energy" proposal [11].

The dynamical nature of dark energy introduces a new cosmological problem, namely why are the densities of vacuum energy and dark matter nearly equal today although they scale independently during the expansion history. To solve the problem, one would require that the matter density and dark energy density always approach their current values independent of the initial conditions. The elaboration of this 'coincidence' problem led to the consideration of generalized versions of the aforementioned models with the inclusion of a coupling between dark energy and dark matter. Thus, various forms of "interacting" dark energy models [12, 13] have been constructed in order to fulfil the observational requirements.

One decisive test for dark energy models is the investigation of their phase-space analysis. In particular, to examine whether they possess attractor solutions corresponding to accelerating universes and ratio $\Omega_{\text{dark energy}}/\Omega_{\text{dark matter}}$ of the order 1. If these conditions are fulfilled, then the universe will result to that solution at late times, independently of the initial conditions, and the basic observational requirements will be satisfied. Although non-interacting quintessence [14, 15], non-interacting phantom [16] and non-interacting quintom models [17] admit late-time accelerated attractors, they possess $\Omega_{\text{dark energy}} = 1$ and thus they are unable to solve the coincidence problem.

When dark energy - dark matter interaction is switched on in quintessence, then one can find accelerated attractors which moreover give $\Omega_{\text{dark energy}}/\Omega_{\text{dark matter}} \approx \mathcal{O}(1)$ [18, 19, 20, 21], but paying the price of introducing new problems such as justifying a non-trivial, almost tuned, sequence of cosmological epochs [22]. In cases of interacting phantom models [23], the existing literature remains in some special coupling forms with mainly numerical results, which suggest that the coincidence problem might be alleviated [13, 24].

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In the present work we are interested in performing a detailed phase-space analysis of the interacting phantom paradigm, considering new and general forms of the interaction term Q. In addition, we go beyond the simplified non-local forms of the literature, considering also a local Q-form proportional to a density [21, 25, 26, 27]. Doing so we find that although late-time accelerated attractors do exist in all the models, as it is expected for phantom cosmology, almost all of them correspond to complete dark energy domination and thus they are unable to solve the coincidence problem. Only for a narrow area of the parameter space of one particular model, can the corresponding solution alleviate the aforementioned problem.

The plan of the work is as follows: In section II we construct the interacting phantom cosmological scenario and we present the formalism for its transformation into an autonomous dynamical system, suitable for a phase-space stability analysis. In section III we perform the phase-space analysis for four different models, using various interaction terms, and in section IV we discuss the corresponding cosmological implications. Finally, section V is devoted to the summary of the obtained results.

II. PHANTOM COSMOLOGY

Let us construct the interacting phantom cosmological paradigm. Throughout the work we consider a flat Robertson-Walker metric:

$$ds^2 = dt^2 - a^2(t)d\mathbf{x^2}, \tag{1}$$

with a the scale factor.

The evolution equations for the phantom and dark matter density (considered as dust for simplicity) are:

$$\dot{\rho}_m + 3H\rho_m = -Q \tag{2}$$
$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = Q, \tag{3}$$

with Q a general interaction term and H the Hubble parameter. Therefore, $Q > 0$ corresponds to energy transfer from dark matter to dark energy, while $Q < 0$ corresponds to dark energy transformation to dark matter. For simplicity, we take the usual phantom energy density and pressure:

$$\rho_\phi = -\frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{4}$$

$$p_\phi = -\frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{5}$$

where $V(\phi)$ is the phantom potential. Equivalently, the phantom evolution equation can be written as:

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\partial V(\phi)}{\partial \phi} = -\frac{Q}{\dot{\phi}}. \tag{6}$$

Finally, the system of equation closes by considering the Friedmann equations:

$$H^2 = \frac{\kappa^2}{3}(\rho_\phi + \rho_m), \tag{7}$$

$$\dot{H} = -\frac{\kappa^2}{2}\Big(\rho_\phi + p_\phi + \rho_m\Big), \tag{8}$$

where we have set $\kappa^2 \equiv 8\pi G$. Although we could straightforwardly include baryonic matter and radiation in the model, for simplicity reasons we neglect them.

In phantom cosmological models, the dark energy is attributed to the phantom field, and its equation of state is given by

$$w_\phi = \frac{p_\phi}{\rho_\phi}. \tag{9}$$

Alternatively one could construct the equivalent uncoupled model described by:

$$\dot{\rho}_m + 3H(1 + w_{m,eff})\rho_m = 0 \tag{10}$$
$$\dot{\rho}_\phi + 3H(1 + w_{\phi,eff})\rho_\phi = 0, \tag{11}$$

where

$$w_{m,eff} = \frac{Q}{3H\rho_m} \tag{12}$$

$$w_{\phi,eff} = w_\phi - \frac{Q}{3H\rho_\phi}. \tag{13}$$

However, it is more convenient to introduce the "total" energy density $\rho_{tot} \equiv \rho_m + \rho_\phi$, obtaining:

$$\dot{\rho}_{tot} + 3H(1 + w_{tot})\rho_{tot} = 0, \tag{14}$$

with

$$w_{tot} = \frac{p_\phi}{\rho_\phi + \rho_m} = w_\phi \Omega_\phi, \tag{15}$$

where $\Omega_\phi \equiv \frac{\rho_\phi}{\rho_{tot}} \equiv \Omega_{\text{dark energy}}$. Obviously, since $\rho_{tot} = 3H^2/\kappa^2$, (14) leads to a scale factor evolution of the form $a(t) \propto t^{2/(3(1+w_{tot}))}$, in the constant w_{tot} case. However, in the late-time stationary solutions that we are studying in the present work, w_{tot} has reached to a constant value and thus the above behavior is valid. Therefore, we conclude that in such stationary solutions the condition for acceleration is just $w_{tot} < -1/3$.

In order to perform the phase-space and stability analysis of the phantom model at hand, we have to transform the aforementioned dynamical system into its autonomous form [14, 15]. This will be achieved by introducing the auxiliary variables:

$$x = \frac{\kappa\dot{\phi}}{\sqrt{6}H},$$
$$y = \frac{\kappa\sqrt{V(\phi)}}{\sqrt{3}H}, \tag{16}$$

together with $M = \ln a$. Thus, it is easy to see that for every quantity F we acquire $\dot{F} = H\frac{dF}{dM}$. Using these variables we obtain:

$$\Omega_\phi \equiv \frac{\kappa^2\rho_\phi}{3H^2} = -x^2 + y^2, \tag{17}$$

$$w_\phi = \frac{-x^2 - y^2}{-x^2 + y^2}, \tag{18}$$

and

$$w_{tot} = -x^2 - y^2. \tag{19}$$

We mention that relations (18) and (19) are always valid, that is independently of the specific state of the system (they are valid in the whole phase-space and not only at the critical points). Finally, note that in the case of complete dark energy domination, i.e $\rho_m \to 0$ and $\Omega_\phi \to 1$, we acquire $w_{tot} \approx w_\phi < -1$, as expected to happen in phantom-dominated cosmology.

The next step is the introduction of a specific ansatz for the interaction term Q. In this case the equations of motion (2), (6), (7) and (8) can be transformed to an autonomous system containing the variables x and y and their derivatives with respect to $M = \ln a$. The consideration of various Q-ansatzes is performed in the next section.

A final assumption must be made in order to handle the potential derivative that is present in (6). The usual assumption in the literature is to assume an exponential potential of the form

$$V = V_0 \exp(-\kappa\lambda\phi), \tag{20}$$

since exponential potentials are known to be significant in various cosmological models [14, 15]. Note that equivalently, but more generally, we could consider potentials satisfying $\lambda = -\frac{1}{\kappa V(\phi)}\frac{\partial V(\phi)}{\partial \phi} \approx const$ (for example this relation is valid for arbitrary but nearly flat potentials [28]).

Having transformed the cosmological system into its autonomous form:

$$\mathbf{X}' = \mathbf{f}(\mathbf{X}), \tag{21}$$

where $\mathbf{X}$ is the column vector constituted by the auxiliary variables, $\mathbf{f(X)}$ the corresponding column vector of the autonomous equations, and prime denotes derivative with respect to $M = \ln a$, we extract its critical points $\mathbf{X_c}$ satisfying $\mathbf{X'} = \mathbf{0}$. Then, in order to determine the stability properties of these critical points, we expand (21) around $\mathbf{X_c}$, setting $\mathbf{X} = \mathbf{X_c} + \mathbf{U}$ with $\mathbf{U}$ the perturbations of the variables considered as a column vector. Thus, for each critical point we expand the equations for the perturbations up to the first order as:

$$\mathbf{U'} = \mathbf{\Xi} \cdot \mathbf{U}, \tag{22}$$

where the matrix $\mathbf{\Xi}$ contains the coefficients of the perturbation equations. Thus, for each critical point, the eigenvalues of $\mathbf{\Xi}$ determine its type and stability.

III. PHASE-SPACE ANALYSIS

In the previous section we constructed the interacting phantom cosmological model, with an arbitrary interaction term Q, and we presented the formalism for its transformation into an autonomous dynamical system, suitable for a phase-space stability analysis. In this section we introduce four specific forms for Q and we perform a complete phase-space analysis.

A. Interacting model 1

This specific interacting model is characterized by a coupling of the form [20, 21, 29]:

$$Q = \alpha H \rho_m. \tag{23}$$

Thus, inserting the auxiliary variables (16) into the equations of motion (2), (6), (7) and (8), and using (23) we result in the following autonomous system:

$$x' = -3x + \frac{3}{2}x(1 - x^2 - y^2) - \sqrt{\frac{3}{2}}\lambda y^2 - \frac{\alpha}{2x}(1 + x^2 - y^2)$$

$$y' = \frac{3}{2}y(1 - x^2 - y^2) - \sqrt{\frac{3}{2}}\lambda xy. \tag{24}$$

The critical points (x_c, y_c) of the autonomous system (24) are obtained by setting the left hand sides of the equations to zero. The real and physically meaningful (i.e corresponding to $y > 0$ and $0 \leq \Omega_\phi \leq 1$) of them are presented in table I. In the same table we present the necessary conditions for their existence.

The 2×2 matrix $\mathbf{\Xi}$ of the linearized perturbation equations writes:

$$\mathbf{\Xi} = \begin{bmatrix} 3\left[-1 + \frac{(1-x_c^2-y_c^2)}{2} - x_c^2 - \frac{\alpha}{3} + \frac{\alpha}{6x_c^2}(1 + x_c^2 - y_c^2)\right] & y_c\left[-\sqrt{6}\lambda - 3x_c + \frac{\alpha}{x_c}\right] \\ y_c\left[-\sqrt{\frac{3}{2}}\lambda - 3x_c\right] & \left[\frac{3(1-x_c^2-y_c^2)}{2} - 3y_c^2 - \sqrt{\frac{3}{2}}\lambda x_c\right] \end{bmatrix}.$$

Therefore, for each critical point of table I, we examine the sign of the real part of the eigenvalues of $\mathbf{\Xi}$, which determines the type and stability of this specific critical point. In table I we present the results of the stability analysis. In addition, for each critical point we calculate the values of w_{tot} (given by relation (19)), and of Ω_ϕ (given by (17)). Thus, knowing w_{tot} we can express the acceleration condition $w_{tot} < -1/3$ in terms of the model parameters.

In order to present the results more transparently, in fig. 1 we depict the range of the $2D$ parameter space (α, λ) that corresponds to an existing and stable critical point B of table I, i.e to negative real parts of the corresponding eigenvalues. Furthermore, we numerically evolve the autonomous system (24) for the parameters $\alpha = 0.5$, $\lambda = 1.0$ and $\alpha = -3.1$, $\lambda = 0.1$, and the results are shown in figure 2. Depending on which region of the parameter-space do the chosen parameter-pair belong (see table (I)), the system lies in the basin of attraction of either the critical point A or B, and thus it is attracted by the one or the other. This is apparent in figure 2, where we present two different parameter choices, leading respectively to point B and A.

B. Interacting model 2

In this model we consider an interaction term of the form [20]:

$$Q = \alpha_0 \kappa^{2n} H^{3-2n} \rho_m^n. \tag{25}$$

Cr. P.	x_c	y_c	Existence	Stable for	Ω_ϕ	w_{tot}	Acceleration
A	$-\frac{\lambda}{\sqrt{6}}$	$\sqrt{1+\frac{\lambda^2}{6}}$	all α,λ	$-\alpha < \lambda^2 + 3$	1	$-1-\frac{\lambda^2}{3}$	all α,λ
B	$\frac{3+\alpha}{\sqrt{6}\lambda}$	$\sqrt{-\frac{\alpha}{3}-\frac{(3+\alpha)^2}{6\lambda^2}}$	$\lambda^2 \geq \frac{(3+\alpha)^2}{-\alpha}$ for $0 \geq \alpha \geq -3$ $-(3+\alpha) \geq \lambda^2 \geq \frac{(3+\alpha)^2}{-\alpha}$ for $\alpha \leq -3$	$\alpha < -3$ $-(3+\alpha) > \lambda^2 \geq \frac{(3+\alpha)^2}{-\alpha}$	$-\frac{(3+\alpha)^2}{3\lambda^2}-\frac{\alpha}{3}$	$\frac{\alpha}{3}$	$\alpha < -1$

TABLE I: The real and physically meaningful critical points of interacting model 1 and their behavior.
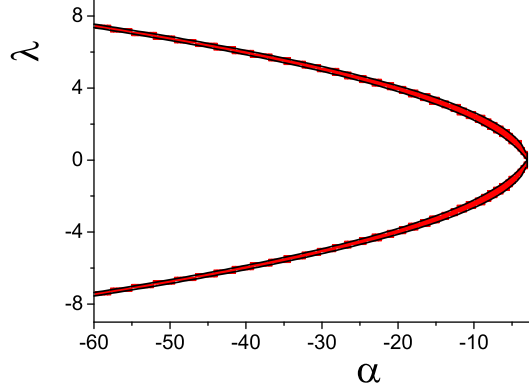

FIG. 1: *The range of the 2D parameter space (α,λ) that corresponds to an existing and stable critical point B of table I, that is in the case of interacting model 1.*
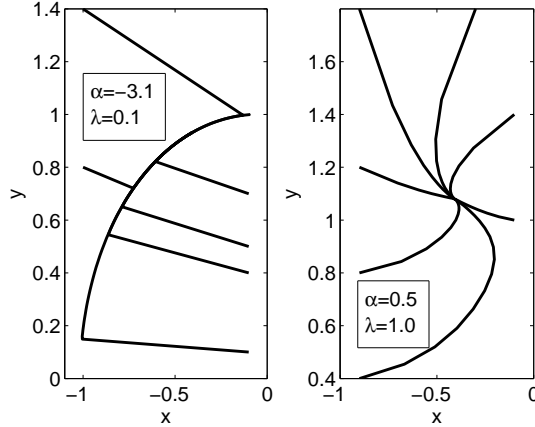

FIG. 2: Phase-space trajectories for interacting model 1. The left panel corresponds to $\alpha = -3.1$ and $\lambda = 0.1$, and thus to the stable fixed point B of table I with $(x_c, y_c)=$(-0.41, 0.93). The right panel corresponds to $\alpha = 0.5$ and $\lambda = 1.0$ and thus to the stable fixed point A of table I with $(x_c, y_c)=$(-0.41, 1.08).

This ansatz corresponds to a generalization of interacting model 1. In the following, we take $n = 2$ for simplicity. Inserting (25) into the cosmological equations (2), (6), (7) and (8), using the auxiliary variables, we acquire the following autonomous system:

$$x' = -3x - \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x(1 - x^2 - y^2) - \frac{3}{2}\alpha_0 \frac{(1 + x^2 - y^2)^2}{x}$$

$$y' = -\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y(1 - x^2 - y^2). \tag{26}$$

The real and physically meaningful critical points (x_c, y_c) of the autonomous system (26) are:

$$\left(x_{c1} = \pm\sqrt{\frac{-\alpha_0}{1+\alpha_0}},\ y_{c1} = 0\right), \quad \left(x_{c2} = -\frac{\lambda}{\sqrt{6}},\ y_{c2} = \sqrt{1+\frac{\lambda^2}{6}}\right),$$

$$\left(x_{c3},\ y_{c3} = \sqrt{1-x_{c3}^2 - \sqrt{6}\lambda x_{c3}/3}\right), \quad \left(x_{c4},\ y_{c4} = \sqrt{1 - x_{c4}^2 - \sqrt{6}\lambda x_{c4}/3}\right),$$

(27)

where

$$x_{c3} = \frac{-(\alpha_0 - 1)\lambda + \sqrt{\lambda^2(\alpha_0 - 1)^2 - 12\alpha_0}}{2\sqrt{6}\alpha_0},$$

(28)

and

$$x_{c4} = \frac{-(\alpha_0 - 1)\lambda - \sqrt{\lambda^2(\alpha_0 - 1)^2 - 12\alpha_0}}{2\sqrt{6}\alpha_0}.$$

(29)

These critical points are presented in table II. The matrix $\mathbf{\Xi}$ of the linearized perturbation equations is:

$$\mathbf{\Xi} = \begin{bmatrix} 3\left[-\frac{(1+3x_c^2+y_c^2)}{2} - 2\alpha_0(1+x_c^2-y_c^2) + \frac{\alpha_0}{2}\frac{(1+x_c^2-y_c^2)^2}{x_c^2}\right] & y_c\left[-\sqrt{6}\lambda - 3x_c + 6\alpha_0\frac{1+x_c^2-y_c^2}{x_c}\right] \\ y_c\left[-\sqrt{\frac{3}{2}}\lambda - 3x_c\right] & \left[\frac{3(1-x_c^2-y_c^2)}{2} - 3y_c^2 - \sqrt{\frac{3}{2}}\lambda x_c\right] \end{bmatrix}.$$

Examining the eigenvalues of the matrix $\mathbf{\Xi}$ for each critical point, we determine its stability behavior. Finally, in table II we also present w_{tot}, Ω_ϕ and the acceleration condition. Note that the values of w_{tot} and Ω_ϕ for the critical points C and D can be straightforwardly calculated using (27) and (28), (29), but we do not present them since these points are unstable and thus non-relevant for our discussion.

Cr. P.	x_c	y_c	Existence	Stable for	Ω_ϕ	w_{tot}	Acceleration
A	$\pm\sqrt{-\frac{\alpha_0}{1+\alpha_0}}$	0	$\alpha_0 = 0$	Unstable	0	0	No
B	$-\frac{\lambda}{\sqrt{6}}$	$\sqrt{1+\frac{\lambda^2}{6}}$	all λ, α_0	all λ, α_0	1	$-1-\frac{\lambda^2}{3}$	all λ, α_0
C	x_{c3}	y_{c3}	$\lambda \le -\sqrt{\frac{3(1+\alpha_0)^2}{-\alpha_0}},\ -1 \le \alpha_0 < 0$ $\lambda \le \sqrt{\frac{3(1+\alpha_0)^2}{-\alpha_0}},\ \alpha_0 < -1$	Unstable	$-x_{c3}^2 + y_{c3}^2$	$-x_{c3}^2 - y_{c3}^2$	$-\sqrt{\frac{-4\alpha_0}{1+2\alpha_0}} < \lambda \le -\sqrt{\frac{3(1+\alpha_0)^2}{-\alpha_0}},\ -1/2 < \alpha_0 < 0$ $\lambda \le -\sqrt{\frac{3(1+\alpha_0)^2}{-\alpha_0}},\ -1 \le \alpha_0 \le -1/2$ $\lambda \le \sqrt{\frac{3(1+\alpha_0)^2}{-\alpha_0}},\ \alpha_0 < -1$
D	x_{c4}	y_{c4}	$\lambda \ge \sqrt{\frac{3(1+\alpha_0)^2}{-\alpha_0}},\ -1 \le \alpha_0 < 0$ $\lambda \ge -\sqrt{\frac{3(1+\alpha_0)^2}{-\alpha_0}},\ \alpha_0 < -1$	Unstable	$-x_{c4}^2 + y_{c4}^2$	$-x_{c4}^2 - y_{c4}^2$	$\sqrt{\frac{-4\alpha_0}{1+2\alpha_0}} > \lambda \ge \sqrt{\frac{3(1+\alpha_0)^2}{-\alpha_0}},\ -1/2 < \alpha_0 < 0$ $\lambda \ge \sqrt{\frac{3(1+\alpha_0)^2}{-\alpha_0}},\ -1 \le \alpha_0 \le -1/2$ $\lambda \ge -\sqrt{\frac{3(1+\alpha_0)^2}{-\alpha_0}},\ \alpha_0 < -1$

TABLE II: The real and physically meaningful critical points of interacting model 2 and their behavior.

In order to make the phase-space analysis more transparent, we elaborate the autonomous system (26) numerically for the parameter values $\alpha_0 = 0.5$ and $\lambda = 1.0$. The corresponding phase-space trajectories, arising imposing different initial conditions, are shown in figure 3. As expected, the system is always attracted by the stable fixed point B of table II.

C. Interacting Model 3

This model is characterized by an interaction term of the form

$$Q = \beta\kappa^{2n}H^{1-2n}\rho_m^n\dot{\phi}^2,$$

(30)

where for simplicity we consider $n = 1$. The autonomous system reads:

$$x' = -3x - \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x(1 - x^2 - y^2) - 3\beta x(1 + x^2 - y^2)$$

$$y' = -\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y(1 - x^2 - y^2).$$
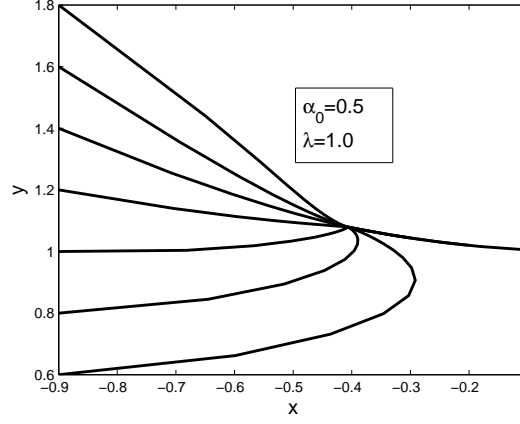
(31)

FIG. 3: Phase-space trajectories for interacting model 2 using $\alpha_0 = 0.5$ and $\lambda = 1.0$. The stable fixed point is the critical point B in table II, with $(x_{c2}, y_{c2})=(-0.41, 1.08)$.

The real and physically meaningful critical points (x_c, y_c) of the autonomous system (31) are:

$$(x_{c1} = 0, \; y_{c1} = 0), \quad (x_{c2} = -\frac{\lambda}{\sqrt{6}}, \; y_{c2} = \sqrt{1 + \frac{\lambda^2}{6}}),$$

$$(x_{c3}, \; y_{c3} = \sqrt{1 - x_{c3}^2 - \sqrt{6}\lambda x_{c3}/3}), \quad (x_{c4}, \; y_{c4} = \sqrt{1 - x_{c4}^2 - \sqrt{6}\lambda x_{c4}/3}), \tag{32}$$

where

$$x_{c3} = \frac{\lambda + \sqrt{\lambda^2 - 12\beta}}{2\sqrt{6}\beta}, \tag{33}$$

and

$$x_{c4} = \frac{\lambda - \sqrt{\lambda^2 - 12\beta}}{2\sqrt{6}\beta}. \tag{34}$$

The matrix Ξ of the linearized perturbation equations is:

$$\Xi = \begin{bmatrix} 3\left[-\frac{(1+3x_c^2+y_c^2)}{2} - \beta(1 + 3x_c^2 - y_c^2)\right] & y_c\left[-\sqrt{6}\lambda - 3x_c + 6\beta x_c\right] \\ y_c\left[-\sqrt{\frac{3}{2}}\lambda - 3x_c\right] & \left[\frac{3(1-x_c^2-y_c^2)}{2} - 3y_c^2 - \sqrt{\frac{3}{2}}\lambda x_c\right] \end{bmatrix}.$$

The critical points are presented in table III, together with their stability behavior, w_{tot}, Ω_ϕ and the acceleration condition.

Cr. P.	x_c	y_c	Existence	Stable for	Ω_ϕ	w_{tot}	Acceleration
A	0	0	all β, λ	Unstable	0	0	No
B	$-\frac{\lambda}{\sqrt{6}}$	$\sqrt{1+\frac{\lambda^2}{6}}$	all β, λ	all λ, $\beta \geq -1$ $\lambda^2 < -\frac{3}{1+\beta}$, $\beta < -1$	1	$-1 - \frac{\lambda^2}{3}$	all β, λ
C	x_{c3}	y_{c3}	$\lambda \leq \sqrt{\frac{-3}{1+\beta}}$, $\beta < -1$	Unstable	$-x_{c3}^2 + y_{c3}^2$	$-x_{c3}^2 - y_{c3}^2$	$-\sqrt{-4\beta} < \lambda \leq \sqrt{\frac{-3}{1+\beta}}$, $\beta < -1$
D	x_{c4}	y_{c4}	$-\sqrt{\frac{-3}{1+\beta}} \leq \lambda$, $\beta < -1$	Unstable	$-x_{c4}^2 + y_{c4}^2$	$-x_{c4}^2 - y_{c4}^2$	$-\sqrt{\frac{-3}{1+\beta}} \leq \lambda < \sqrt{-4\beta}$, $\beta < -1$

TABLE III: The real and physically meaningful critical points of interacting model 3 and their behavior.

In order to present the results more transparently, in fig. 4 we depict the range of the $2D$ parameter space (β, λ) that corresponds to the existence of the stable critical point B of table III. Furthermore, we elaborate the autonomous system (31) numerically, for the parameter values $\lambda = 1.0$, $\beta = 0.5$ and $\lambda = 1.0$, $\beta = -2.0$. The phase-space trajectories, corresponding to different initial conditions, are shown in figure 5. In these cases, the system is always attracted by the stable fixed point B of table III, as expected.
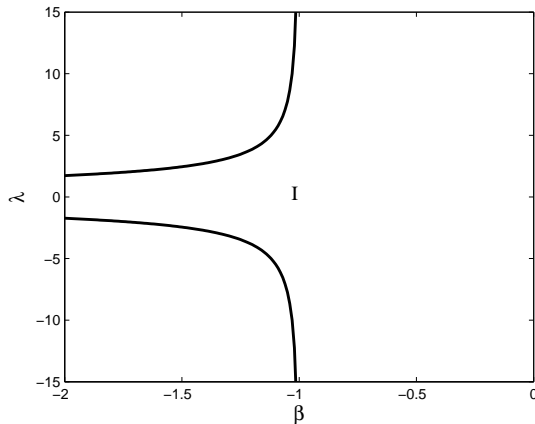
FIG. 4: The range of the $2D$ parameter space (β, λ) that corresponds to an existing and stable critical point B of table III, that is in the case of interacting model 3.
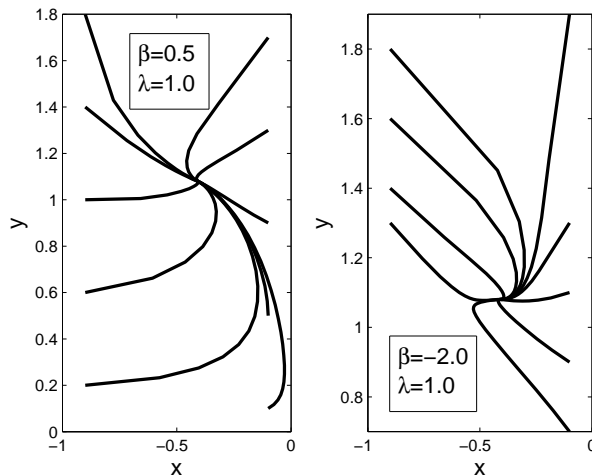

FIG. 5: Phase-space trajectories for interacting model 3 with $\beta = 0.5$, $\lambda = 1.0$ (left panel) and $\beta = -2.0$, $\lambda = 1.0$ (right panel). In both cases the stable fixed point is the critical point B of table III, with $(x_{c2}, y_{c2})=$(-0.41, 1.08).

D.   Interacting Model 4

In the previous subsections we considered interaction terms depending explicitly on the Hubble parameter H, that is interaction terms determined by the properties of the whole universe. In the present model we consider the local interaction term [21]

$$Q = \Gamma \rho_m, \tag{35}$$

where we assume that the coefficient Γ is a constant. Note that this ansatz, with $\Gamma > 0$, has been used in different frameworks to describe the decay of dark matter into radiation [25], the decay of a curvaton field into radiation [26] and the decay of superheavy dark matter particles into a quintessence scalar field [27].

The complexity of the interaction term (35), comparing to the previous ones, is revealed by the fact that the dynamical evolution cannot be elaborated using only the two auxiliary variables (16). Indeed, H cannot be eliminated from the autonomous equations. In order to achieve that we introduce an additional auxiliary variable as

$$v = \frac{H_0}{H}, \tag{36}$$

where H_0 is a constant. Thus, using (35), the cosmological equations (2), (6), (7) and (8) are transformed to the

following autonomous system:

$$x' = -3x + \frac{3}{2}x(1 - x^2 - y^2) - \sqrt{\frac{3}{2}}\lambda y^2 - \frac{\gamma v}{2x}(1 + x^2 - y^2)$$

$$y' = \frac{3}{2}y(1 - x^2 - y^2) - \sqrt{\frac{3}{2}}\lambda xy$$

$$v' = \frac{3}{2}v(1 - x^2 - y^2), \tag{37}$$

where we have introduced the dimensionless coupling constant

$$\gamma = \frac{\Gamma}{H_0}. \tag{38}$$

The 3×3 matrix $\boldsymbol{\Xi}$ of the linearized perturbation equations is:

$$\boldsymbol{\Xi} = \begin{bmatrix} 3\left[-\frac{(1+3x_c^2+y_c^2)}{2} + \frac{\gamma v_c}{6x_c^2}(1 + x_c^2 - y_c^2) - \frac{\gamma v_c}{3}\right] & y_c\left[-\sqrt{6}\lambda - 3x_c + \frac{\gamma v_c}{x_c}\right] & -\frac{\gamma}{2x_c}(1 + x_c^2 - y_c^2) \\ y_c\left[-\sqrt{\frac{3}{2}}\lambda - 3x_c\right] & \left[\frac{3(1-x_c^2-y_c^2)}{2} - 3y_c^2 - \sqrt{\frac{3}{2}}\lambda x_c\right] & 0 \\ -3x_c v_c & -3y_c v_c & \frac{3}{2}(1 - x_c^2 - y_c^2) \end{bmatrix}.$$

In this model there is only one real and physically meaningful critical point presented in table IV, together with its stability behavior, w_{tot}, Ω_ϕ and the acceleration condition.

Cr. P.	x_c	y_c	v_c	Existence	Stable for	Ω_ϕ	w_{tot}	Acceleration
A	$-\frac{\lambda}{\sqrt{6}}$	$\sqrt{1 + \frac{\lambda^2}{6}}$	0	all γ,λ	all γ,λ	1	$-1 - \frac{\lambda^2}{3}$	all γ,λ

TABLE IV: The real and physically meaningful critical point of interacting model 4 and its behavior.
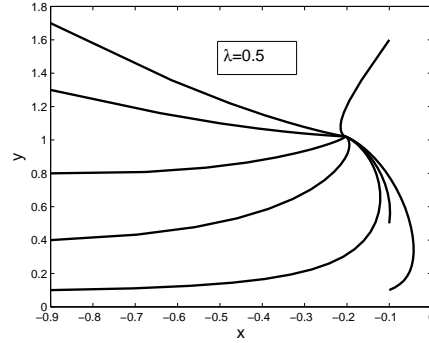

FIG. 6: Phase-space trajectories for interacting model 4 with $\lambda = 0.5$. The stable fixed points is the critical point A of table IV, with (x_c, y_c)=(-0.20, 1.02).

To reveal this behavior more transparently, we solve numerically the autonomous system (37) for $\lambda = 0.5$. In order to plot the two dimensional phase trajectories for the variables x and y, we project the system onto the $v = 0$ plane and the results are shown in figure 6. Finally, we mention that for the corresponding model of a canonical field, examined in [21], using the variable $z = H_0/(H_0 + H)$ the authors found "critical points" for $z = 1$, which corresponds to $v = \infty$ in our case. However, as we can see, this point is not a critical point of the system.

IV.   COSMOLOGICAL IMPLICATIONS

Since we have performed a complete phase-space analysis of various interacting phantom models, we can now discuss the corresponding cosmological behavior. A general remark is that this behavior is radically different from the

corresponding interacting quintessence models with the same couplings [20, 21]. Furthermore, a common feature of all the examined models is that we obtain $w_\phi < -1$ in the whole phase-space, and $w_{tot} < -1$ for all the existing and stable critical points. Thus, independently of the specific interacting model and of the chosen initial conditions, we will be always moving on the same side (below) of the phantom divide. This was expected for phantom cosmology. However, we remind that interacting quintessence can conditionally describe the -1-crossing, in contrast to expectations [18, 19, 20, 21]. Finally, we mention that the fact that w_{tot} is not only less than $-1/3$, as required by the acceleration condition, but it is always less than -1 in all stable solutions, leads to $\dot{H} > 0$ at all times. Thus, these solutions correspond to a super-accelerating universe [30], that is with a permanently increasing H. If $H \to \infty$ at $t \to \infty$ then we acquire an eternally expanding universe, while if $H \to \infty$ at $t \to t_{BR} < \infty$ then the universe results to a Big Rip. These behaviors are also common in phantom cosmology [7, 31].

A. Interacting Model 1

In this model, both critical points A and B are stable points, and thus late-time attractors (see table I). Point A, which is stable for a large region of the $2D$ parameter space (α, λ), corresponds to an accelerating universe. In addition, since $\Omega_\phi = 1$ it corresponds to a complete dark energy domination. Note that both these features hold independently of the α-value if it is positive, which was expected since positive α means positive Q, that is energy transfer from dark matter to dark energy. So it leads to a final, complete dark energy domination. In conclusion, we see that the stable fixed point A does correspond to an accelerating universe, but cannot solve the coincidence problem since it leads to $\Omega_{\text{dark energy}} = 1$ instead of $\Omega_{\text{dark energy}}/\Omega_{\text{dark matter}} \approx \mathcal{O}(1)$.

On the other hand, the critical point B, for $\alpha < -3$ and for λ-values that make it stable (note that in this case the existence and stability conditions coincide) corresponds to an accelerating universe with $0 < \Omega_\phi < 1$, i.e with $\Omega_{\text{dark energy}}/\Omega_{\text{dark matter}} \approx \mathcal{O}(1)$. Therefore, for this region of the (α, λ) parameter space, the stable fixed point B can satisfy the observational requirements, and at late times the universe is attracted by an accelerating solution with vacuum and matter densities of the same order. For example, if we take $\alpha = -3.1$ and $\lambda = 0.1$ (see left panel of fig. 2), then we obtain $\Omega_\phi = 0.7$ and $w_{tot} = -1.03$, which is in agreement with current observations within 95% confidence level [1]. However, as can be seen from figure 1, the stability region of point B, which produces the aforementioned interesting cosmological behavior, is very narrow. Therefore, the late-time attractor A has significantly larger probability to be the universe's late-time solution. In other words, the phantom interacting model at hand can indeed satisfy the cosmological requirements, but for a very narrow area of the parameter space.

B. Interacting Model 2

In this case we observe that the critical points A, C and D are unstable, and therefore they cannot be late-time cosmological solutions. The only relevant critical point is B, which is a stable fixed points for the whole (α_0, λ) parameter space. As can be seen from table II, it corresponds to an accelerating universe with $\Omega_\phi = 1$, i.e., to a complete dark-energy domination. Thus, it cannot act as a candidate for solving the coincidence problem.

C. Interacting Model 3

In this model we see that the critical points A, C and D are unstable and thus non-relevant. Critical point B is the only cosmologically interesting one, since it corresponds to a late-time attractor for the region of the (β, λ) parameter space shown in figure 4. However, although for that region it always gives rise to acceleration, it possesses $\Omega_\phi = 1$. Therefore it cannot solve the coincidence problem.

D. Interacting Model 4

In this case, the critical point A is the only real and physically meaningful critical point, and it is stable for the whole $2D$ parameter space (γ, λ). It corresponds to an accelerating universe, completely dominated by dark energy $(\Omega_\phi = 1)$. Thus this phantom interacting model cannot solve the coincidence problem.

V.   CONCLUSIONS

In this work we performed a detailed phase-space analysis of various phantom cosmological models, where the dark energy sector interacts with the dark matter one. Our basic goal was to examine whether there exist late-time scaling attractors, corresponding to accelerated universe and possessing $\Omega_{\text{dark energy}}/\Omega_{\text{dark matter}} \approx \mathcal{O}(1)$, thus satisfying the basic observational requirements. We investigated four different interaction models, including one with a local, i.e. H-independent, form of the interaction term (interacting model 4). We extracted the critical points, determined their stabilities, and calculated the basic cosmological observables, namely the total equation-of-state parameter w_{tot} and $\Omega_{\text{dark energy}}$ (attributed to the phantom field). The key point of solving the coincidence problem is that the universe is in the attractor now and it has normal radiation dominated and matter dominated eras before it reached the attractors. Note that as long as the interaction term is not too strong, the standard cosmology can be always recovered. Once it is in the attractor, the results do not depend on the initial conditions. Thus, one can switch on the interaction and consider as initial conditions the end of the known epochs of standard Big Bang cosmology, in order to avoid disastrous interference.

In all the examined models we found that stable late-time solutions do exist, corresponding moreover to an accelerating universe. This feature was expected since phantom cosmology has been constructed in order to always satisfy this condition. Indeed, for all the studied models, we did not find any non-accelerating stable solution. However, in almost all the cases the late-time solutions correspond to a complete dark energy domination and thus are unable to solve the coincidence problem. The only case in which this is possible is in interacting model 1, if we select the parameter values from a very narrow region of the $2D$ parameter space (see figure 1).

In conclusion, we see that the examined models of interacting phantom cosmology can produce acceleration (which is "embedded" in phantom cosmology in general) but cannot solve the coincidence problem, unless one imposes a form of fine-tuning in the model parameters. This result has been extracted by the negative-kinetic-energy realization of phantom, which does not cover the whole class of phantom models, but since it is a qualitative statement it should intuitively be robust for general interacting phantom scenarios, too. An alternative direction would be to consider more complicated interaction terms, suitably constructed in order to solve the coincidence problem. But the interaction term was introduced in phantom cosmology in order to solve the coincidence problem in a simple and general way, avoiding the assumptions and fine-tunings of conventional cosmology. Although promising and with many advantages, interacting phantom cosmology needs further investigation.

[1] A.G. Riess *et al.* [Supernova Search Team Collaboration], Astron. J. **116**, 1009 (1998); A. G. Riess *et al.* [Supernova Search Team Collaboration], Astrophys. J. **607**, 665 (2004); S. Perlmutter *et al.* [Supernova Cosmology Project Collaboration], Astrophys. J. **517**, 565 (1999); D. N. Spergel *et al.*, Astrophys. J. Suppl. **148**, 175 (2003); S. W. Allen, *et al.*, Mon. Not. Roy. Astron. Soc. **353**, 457 (2004).

[2] S.M. Carroll, V. Duvvuri, M. Trodden, M.S. Turner, Phys. Rev. D **70**, 043528 (2004); T. Chiba, Phys. Lett. B **575**, 1 (2003); S. Nojiri, S.D. Odintsov, Phys. Rev. D **68**, 123512 (2003); C.G. Shao, R.G. Cai, B. Wang, R.K. Su, Phys. Lett. B **633**, 164 (2006).

[3] S. Capozziello, Int. J. Mod. Phys. D **11**, 483 (2002); S. A. Appleby and R. A. Battye, Phys. Lett. B **654**, 7 (2007); S. Nojiri, S.D. Odintsov, Int. J. Geom. Meth. Mod. Phys. **4**, 115 (2007); A. A. Starobinsky, JETP Lett. **86**, 157 (2007); W. Hu, I. Sawicki, Phys. Rev. D **76**, 064004 (2007); S. Nojiri, S.D. Odintsov, [arXiv:0807.0685[hep-th]].

[4] G.R. Dvali, G. Gabadadze, M. Porrati, Phys. Lett. B **485**, 208 (2000); C. Deffayet, G.R. Dvali, G. Gabadadze, Phys. Rev. D **65**, 044023 (2002); Y.G. Gong, C.K. Duan, Class. Quantum Grav. **21**, 3655 (2004); F. K. Diakonos and E. N. Saridakis, JCAP **0902**, 030 (2009); Y.G. Gong, C.K. Duan, Mon. Not. Roy. Astron. Soc. **352**, 847 (2004); Y.G. Gong, [arXiv:0808.1316[astro-ph]].

[5] P. Binétruy, C. Deffayet, D. Langlois, Nucl. Phys. B **565**, 269 (2000); R.G. Cai, Y.G. Gong, B. Wang, JCAP **0603**, 006 (2006); Y.G. Gong, A. Wang, Class. Quantum Grav. **23**, 3419 (2006); Y.G. Gong, A. Wang, Q. Wu, Phys. Lett. B **663**, 147 (2008); M. R. Setare and E. N. Saridakis, Phys. Lett. B **670**, 1 (2008); M. R. Setare and E. N. Saridakis, JCAP **0903**, 002 (2009).

[6] B. Ratra and P. J. E. Peebles, Phys. Rev. D **37**, 3406 (1988); C. Wetterich, Nucl. Phys. B **302**, 668 (1988); A. R. Liddle and R. J. Scherrer, Phys. Rev. D **59**, 023509 (1998); I. Zlatev, L. M. Wang and P. J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999); Z. K. Guo, N. Ohta and Y. Z. Zhang, Mod. Phys. Lett. A **22**, 883 (2007).

[7] R. R. Caldwell, Phys. Lett. B **545**, 23 (2002); R. R. Caldwell, M. Kamionkowski and N. N. Weinberg, Phys. Rev. Lett. **91**, 071301 (2003); S. Nojiri and S. D. Odintsov, Phys. Lett. B **562**, 147 (2003); V. K. Onemli and R. P. Woodard, Phys. Rev. D **70**, 107301 (2004) [arXiv:gr-qc/0406098]; M. R. Setare, Eur. Phys. J. C **50**, 991 (2007); E. N. Saridakis, [arXiv:0811.1333 [hep-th]].

[8] B. Boisseau, G. Esposito-Farese, D. Polarski and A. A. Starobinsky, Phys. Rev. Lett. **85**, 2236 (2000); S. Nojiri, S. D. Odintsov and M. Sasaki, Phys. Rev. D **71**, 123509 (2005); M. z. Li, B. Feng and X. m. Zhang, JCAP **0512**,

002 (2005); S. Nojiri and S. D. Odintsov, Phys. Rev. D **72**, 023003 (2005); S. Sur and S. Das, JCAP **0901**, 007 (2009); K. Bamba, C. Q. Geng, S. Nojiri and S. D. Odintsov, arXiv:0810.4296 [hep-th].

[9] C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, Phys. Rev. D **63**, 103510 (2001) [arXiv:astro-ph/0006373].

[10] B. Feng, X. L. Wang and X. M. Zhang, Phys. Lett. B **607**, 35 (2005); Z. K. Guo, *et al.*, Phys. Lett. B **608**, 177 (2005); M.-Z Li, B. Feng, X.-M Zhang, JCAP, **0512**, 002 (2005); B. Feng, M. Li, Y.-S. Piao and X. Zhang, Phys. Lett. B **634**, 101 (2006); M. R. Setare, Phys. Lett. B **641**, 130 (2006); W. Zhao and Y. Zhang, Phys. Rev. D **73**, 123509 (2006); M. R. Setare and E. N. Saridakis, Phys. Lett. B **671**, 331 (2009).

[11] A. G. Cohen, D. B. Kaplan and A. E. Nelson, Phys. Rev. Lett. **82**, 4971 (1999); P. Horava and D. Minic, Phys. Rev. Lett. **85**, 1610 (2000); S. D. H. Hsu, Phys. Lett. B **594**, 13 (2004); M. Li, Phys. Lett. B **603**, 1 (2004); Y.G. Gong, Phys. Rev. D **70**, 064029 (2004); Y.G. Gong and J. Liu, JCAP **0809**, 010 (2008); D. Pavon and W. Zimdahl, Phys. Lett. B **628**, 206 (2005); H. Li, Z. K. Guo and Y. Z. Zhang, Int. J. Mod. Phys. D **15**, 869 (2006); M. R. Setare, J. Zhang and X. Zhang, JCAP **0703**, 007 (2007); E. N. Saridakis, Phys. Lett. B **660**, 138 (2008); E. N. Saridakis, JCAP **0804**, 020 (2008); E. N. Saridakis, Phys. Lett. B **661**, 335 (2008).

[12] A. P. Billyard and A. A. Coley, Phys. Rev D **61**, 083503 (2000); J. P. Mimoso, A. Nunes and D.Pavon, Phys. Rev. D **73**, 023502 (2006); R. Lazkoz and G. Leon, Phys. Lett. B **638**, 303 (2006); T. Gonzalez, G. Leon and I. Quiros, Class. Quant. Grav. **23**, 3165 (2006); G. R. Farrar and P. J. E. Peebles, Astrophys. J. **604**, 1 (2004); B. Wang, Y. G. Gong and E. Abdalla, Phys. Lett. B **624**, 141 (2005); M. R. Setare, Phys. Lett. B **642**, 1 (2006).

[13] Z. K. Guo, R. G. Cai and Y. Z. Zhang, JCAP **0505**, 002 (2005); A. Nunes, J.P. Mimoso and T.C. Charters, Phys. Rev. D **63**, 083506 (2001); D.F. Mota and C. van de Bruck, Astron. Astrophys. **421**, 71 (2004); M. Manera and D.F. Mota, Mon. Not. Roy. Astron. Soc. **371**, 1373 (2006); N.J. Nunes and D.F. Mota, Mon. Not. Roy. Astron. Soc. **368**, 751 (2006); J.D. Barrow and T. Clifton, Phys. Rev. D **73**, 103520 (2006); T. Clifton and J.D. Barrow, Phys. Rev. D **73**, 104022 (2006); T. Clifton and J.D. Barrow, Phys. Rev. D **75**, 043515 (2007); M. Jamil and M.A. Rashid, arXiv: 0802.1144; M. Jamil, arXiv: 0810.2896.

[14] P.G. Ferreira, M. Joyce, Phys. Rev. Lett. **79**, 4740 (1997); E.J. Copeland, M. Sami, S. Tsujikawa, Int. J. Mod. Phys. D **15**, 1753 (2006); Y.G. Gong, A. Wang, Y.Z. Zhang, Phys. Lett. B **636**, 286 (2006).

[15] E. J. Copeland, A. R. Liddle and D. Wands, Phys. Rev. D **57**, 4686 (1998).

[16] P. Singh, M. Sami and N. Dadhich, Phys. Rev. D **68**, 023522 (2003); J. G. Hao and X. Z. Li, Phys. Rev. D **70**, 043529 (2004).

[17] H. Wei and S. N. Zhang, Phys. Rev. D **76**, 063005 (2007); M. R. Setare and E. N. Saridakis, JCAP **0809**, 026 (2008); M. R. Setare and E. N. Saridakis, Phys. Lett. B **668**, 177 (2008); M. R. Setare and E. N. Saridakis, [arXiv:0807.3807 [hep-th]].

[18] C. Wetterich, Astron. Astrophys. **301**, 321 (1995); L. Amendola, Phys. Rev. D **60**, 043501 (1999).

[19] H. Garcia-Compean, G. Garcia-Jimenez, O. Obregon and C. Ramirez, JCAP **0807**, 016 (2008).

[20] X. M. Chen and Y.G. Gong, [arXiv:0811.1698[gr-qc]].

[21] C. G. Böhmer, G. Caldera-Cabral, R. Lazkoz and R. Maartens, Phys. Rev. D **78**, 023505 (2008).

[22] L. Amendola, M. Quartin, S. Tsujikawa and I. Waga, Phys. Rev. D **74**, 023525 (2006).

[23] Z. K. Guo and Y. Z. Zhang, Phys. Rev. D **71**, 023501 (2005). T. Gonzalez and I. Quiros, Class. Quant. Grav. **25**, 175019 (2008).

[24] R. Curbelo, T. Gonzalez, G. Leon and I. Quiros, Class. Quant. Grav. **23**, 1585 (2006).

[25] R. Cen, Astrophys. J. **546**, L77 (2001) [arXiv:astro-ph/0005206]; M. Oguri, K. Takahashi, H. Ohno and K. Kotake, Astrophys. J. **597**, 645 (2003).

[26] K. A. Malik, D. Wands and C. Ungarelli, Phys. Rev. D **67**, 063516 (2003).

[27] H. Ziaeepour, Phys. Rev. D **69**, 063512 (2004).

[28] R. J. Scherrer and A. A. Sen, Phys. Rev. D **77**, 083515 (2008); R. J. Scherrer and A. A. Sen, Phys. Rev. D **78**, 067303 (2008); M. R. Setare and E. N. Saridakis, Phys. Rev. D **79**, 043005 (2009).

[29] L. P. Chimento, A. S. Jakubi, D. Pavon and W. Zimdahl, Phys. Rev. D **67**, 083513 (2003).

[30] S. Das, P. S. Corasaniti and J. Khoury, Phys. Rev. D **73**, 083509 (2006); M. Kaplinghat and A. Rajaraman, Phys. Rev. D **75**, 103504 (2007).

[31] F. Briscese, E. Elizalde, S. Nojiri and S. D. Odintsov, Phys. Lett. B **646**, 105 (2007).